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SPATIAL TWISTED CENTRAL CONFIGURATIONS OF NEWTONIAN 6-BODY PROBLEM

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Abstract. By employing a simple method that relies on known results for the circulant matrix A_1, and by analyzing the eigenvalues and eigenvectors of four circulant matrices ($\tilde{B}$, $\tilde{B}^*$, $\tilde{D}$ and $\tilde{D}^*$), in the twisted configurations of the spatial Newtonian 6-body problem formed by two parallel equilateral triangles with a distance $h > 0$, we demonstrate that when the twist angle $\theta \in [0, 2\pi)$, there is no spatial twisted central configuration with unequal masses located at the vertices of each separate equilateral triangle.

Keywords: Newtonian 6-body problem, central configurations, regular 3-polygons, twist angles, circulant matrix, eigenvalues and eigenvectors.

AMS Subject Classification: 70F10, 70F15.

1. Introduction and main result

Given n positive masses m_k with positions $x_k \in \mathbb{R}^d$, $k = 1, \ldots, n$ and $d \in \mathbb{Z}^+$, we consider the configuration space $(\mathbb{R}^d)^n \setminus \triangle_n$, where

$$\triangle_n = \{x = (x_1, x_2, \ldots, x_n) \,|\, x_j = x_k,\ 1 \leq j \neq k \leq n\}.$$

A central configuration refers to a specific configuration of n masses, where the acceleration vector of each mass is a scalar multiple of the corresponding position vector, shared by all the masses. To be more precise, the definition of a central configuration is given as follows.

Definition 1.1. [11, Page 109] *We say a configuration $q = (q_1, q_2, \ldots, q_n) \in (\mathbb{R}^d)^n \backslash \triangle_n$ is a central configuration at some moment if there exists a constant $\lambda \in \mathbb{R}$ such that*

$$\sum_{\substack{j \neq k \\ 1 \leq j \leq n}} \frac{m_j m_k (q_j - q_k)}{|q_j - q_k|^3} = -\lambda m_k (q_k - x_0), \qquad k = 1, 2, \ldots, n, \tag{1}$$

where the center of masses is $x_0 = [\sum_{1 \leq k \leq n} m_k q_k]/[\sum_{1 \leq k \leq n} m_k]$.

The study of the Newtonian n-body problem places great importance on central configurations. These configurations play a significant role in the analysis of collision orbits, expanding gravitational systems, and the limitations that affect the configurations assumed by the bounded motion of the n-body problem [17, 21].

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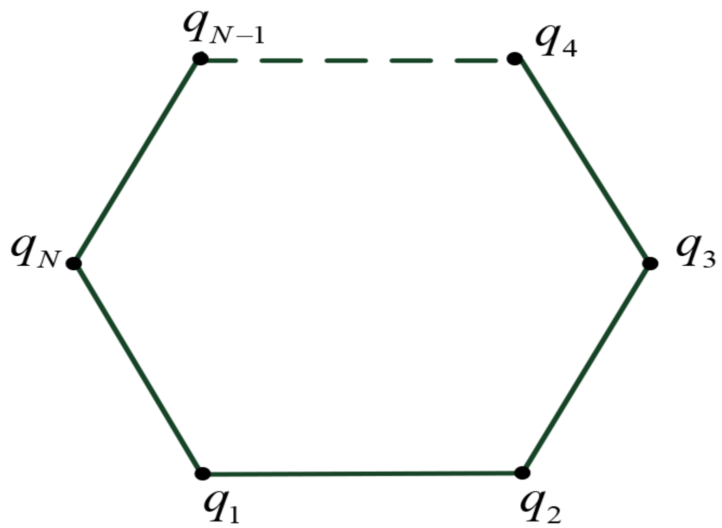

Figure 1. Planar N-body problem.

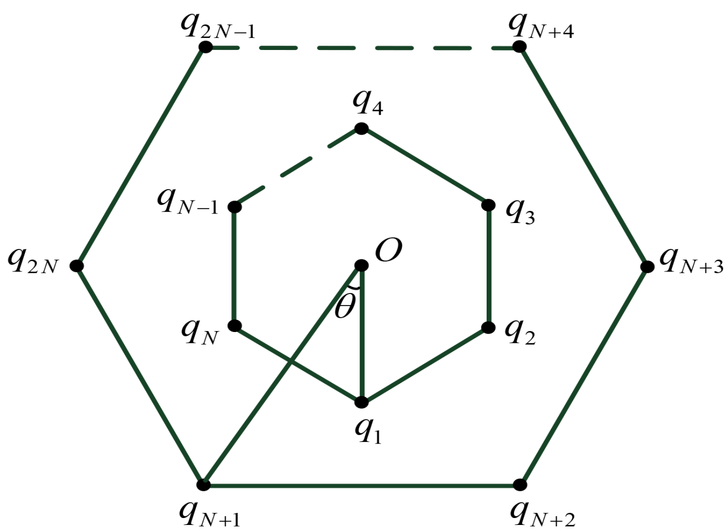

Figure 2. Planar $2N$-body problem with twisted angle

The Newtonian 3-body problem has a well-known central configuration where three masses are positioned at the vertices of a regular 3-polygon. In 1772, Lagrange [9] discovered the Lagrange equilateral-triangle central configuration. For $n = 4$, in [12] the authors made an intriguing finding stating that if four masses are located at the vertices of a regular 4-polygon, they must be equal in weight, which is different from the case of $n = 3$. In [20] the author extended the result to $n = 5$. In 1985, the authors of the paper [16] studied the case where $n = N \geq 4$, and the masses were located at the vertices of a regular N-polygon. They proved the famous result that if the N masses form a central configuration, then the N masses must be equal (refer to Fig.1). Recently, in [1], the authors extended the result of the paper [16] to planar Newtonian $(N+1)$-body problem, and in [19], the author extended the result given in [16] from Newtonian potential to general homogeneous potentials.

Assume that the masses $m_1, \ldots, m_N$ are located at the vertices of one regular N-polygon; the particles $m_{N+1}, \ldots, m_{2N}$ are located at the vertices of the other regular N-polygon (see Fig.3):

$$\begin{cases} q_l = (\rho_l,\ 0), \quad l = 1, \ldots, N, \\ q_s = (r\rho_s e^{i\theta},\ h), \quad s = l + N = N+1, \ldots, 2N, \quad \theta \in [0, 2\pi), \quad h > 0, \end{cases} \tag{2}$$

where $\rho_l = e^{i\theta_l}$ ($\theta_l = 2l\pi/N$) is the l-th root of the N-roots of unity. Then, $|\rho_l| = |\rho_s| = 1$, and we call r and h the ratio of the sizes of the two N-polygons and the distance between the two paralleled regular N-polygons in $\mathbb{R}^3$, respectively.

In 2003, the authors of [24] extended the results given in [16] to planar twisted central configurations of Newtonian $2N$-body problem ($n = 2N \geq 6$), which was formed by two concentric regular N-polygons with any twist angle θ and $h = 0$ (see Fig.2). For this planar central configuration, they showed that the masses in each separate regular N-polygon are equal, and the main tool is based on the analysis of eigenvalues and eigenvectors of the corresponding circulant matrix. For spatial twisted central configurations of Newtonian $2N$-body problem ($n = 2N \geq 6$), since the distance $h > 0$, it is very difficult to analyze the eigenvalues and eigenvectors of the corresponding circulant matrix.

In 2012, the authors of [23] studied spatial twisted central configurations involving two parallel regular N-polygons with distance $h > 0$ (see Fig.3), and under the assumption that the masses in each separate regular N-polygon are equal, they derived that the twist angles must be $\theta = 0$ or $\theta = \pi/N$. In 2015, in order to extend the results given in [24] to spatial twisted central configurations of Newtonian $2N$-body problem (see Fig.3), in [18] the authors introduced a new circulant matrix A_r with $r > 0$, which was to obtain the relationship for eigenvalues of the circulant matrix A_r when the twist angle was $\theta = 0$. They demonstrated that for $k = 1, 2, \ldots, N$, $\lambda_{r,k} > 0$ if $0 < r < 1$ and $\lambda_{r,k} < 0$ if $r > 1$, where r represents the ratio between the radii of the two regular N-polygons. Furthermore, they obtained that for the spatial twisted central configurations, if the twist angle $\theta = 0$, then the masses in each separate regular N-polygon are equal. However, their method only applies when the twist angle is $\theta = 0$, and it doesn't work for twist angles $\theta \in (0, 2\pi)$. For more details in this direction, one can refer to [2, 3, 4, 5, 6, 7, 8, 10, 13, 15, 22].

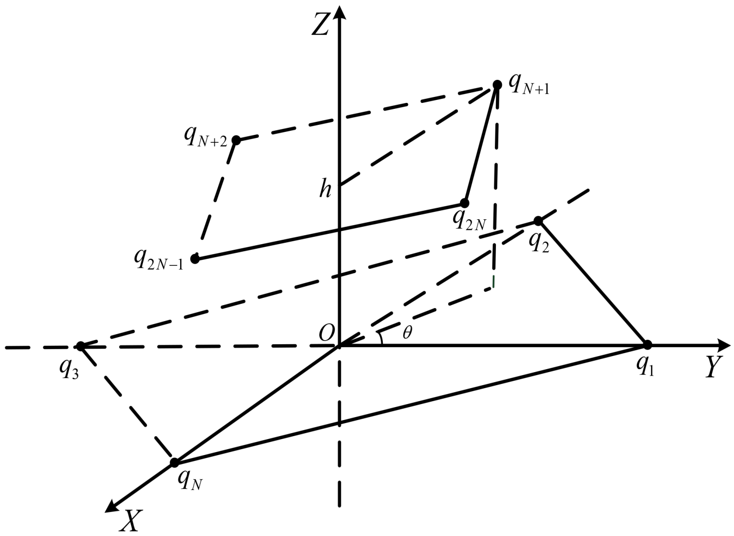

Figure 3. Spatial $2N$-body problem with twisted angle θ

It is important to note that in the Lagrange equilateral-triangle central configuration, where three masses are positioned at the vertices of an equilateral triangle, the three masses are not necessarily equal. Additionally, for the twisted configurations of the spatial Newtonian 6-body problem, which consist of two parallel equilateral triangles with a distance $h > 0$, when the twist angle $\theta = 0$, if the masses at the vertices of each triangle are unequal, then these six masses cannot form a central configuration. This leads to a natural question:

- In the spatial Newtonian 6-body problem formed by two parallel equilateral triangles with a distance $h > 0$, when the twist angles $\theta \in (0, 2\pi)$, does there exist a configuration with unequal masses located at the vertices of each separate equilateral triangle, such that the six masses form a spatial twisted central configuration?

In this paper, by utilizing the relationship between the eigenvalues of four introduced circulant matrices ($\tilde{B}$, $\tilde{B}^*$, $\tilde{D}$, and $\tilde{D}^*$) and some known results of eigenvalues and eigenvectors of the circulant matrix A_1, we can answer the question. In what follows, for six masses $(m_1, m_2, m_3, m_4, m_5, m_6) \in (\mathbb{R}^+)^6$ with positions $q \in (q_1, q_2, q_3, q_4, q_5, q_6) \in (\mathbb{R}^3)^6 \backslash \triangle_6$, in (2), if we set $N = 3$ and $h > 0$, then (2) is translated into

$$\begin{cases} q_l = (\rho_l, 0) = (e^{i\theta_l}, 0) = (e^{\frac{2l\pi}{3}i}, 0), \ l = 1, 2, 3, \\ q_s = (r\rho_s e^{i\theta}, \ h) = (re^{i(\theta_s+\theta)}, \ h) = (re^{i(\frac{2s\pi}{3}+\theta)}, \ h), \ \theta \in [0, 2\pi), \ r > 0, \ h > 0, \ s = 4, 5, 6. \end{cases} \tag{3}$$

Then, our main result is described as follows.

Theorem 1.1. *Let the configuration $q = (q_1, q_2, \cdots, q_6)$ be defined as in (3). If q is a spatial twisted central configuration for the mass vector $m = (m_1, m_2, \cdots, m_6)$, then $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$.*

Employing Theorem 1.1, we have

Corollary 1.1. *For the configuration q in (3) and any $\theta \in [0, 2\pi)$, there is no configuration with unequal masses located at the vertices of each separate equilateral triangle, such that the six masses form a spatial twisted central configuration.*

2. Preliminaries and some useful lemmas

We prove our main result using circulant matrices, and the definition of a circulant matrix is as follows.

Definition 2.1. [14, Pages 65-66] *A matrix $C = (c_{jk})_{3\times 3}\,(1 \leq j, k \leq 3)$ is circulant if $c_{j,k} = c_{j-1,k-1}$ where $c_{j,0} = c_{j,3}$ and $c_{0,k} = c_{3,k}$.*

We have the following lemmas regarding the properties of any circulant matrix.

Lemma 2.1. [16, Page 303] *Every 3×3 circulant matrix has the same forms of the eigenvalues λ_j and the corresponding eigenvectors ν_j, more precisely,*

$$\lambda_j = \sum_{1\leq k\leq 3} c_{1,k}\rho_{j-1}^{k-1}, \quad \nu_j = (\rho_{j-1}, \rho_{j-1}^2, \rho_{j-1}^3)^T, \ \ j = 1, 2, 3. \tag{4}$$

Lemma 2.2. [1, Proposition 2.2] *The eigenvectors ν_j with $j = 1, 2, 3$ form a basis of $\mathbb{C}^3$.*

Lemma 2.3. [14, Page 65] *Let $(\bar{\nu}_k)^T$ be the conjugate transpose of ν_k. Then*

$$(\bar{\nu}_k)^T\nu_j = \begin{cases} 3, & k = j, \\ 0, & k \neq j; \end{cases} \qquad (\rho_2, \rho_4, \rho_6)(\bar{\nu}_3)^T = 3.$$

Let matrices A_1, $\tilde{B}$, $\tilde{B}^*$, $\tilde{D}$ and $\tilde{D}^*$ be defined as follows.

$$A_1 = \begin{pmatrix} 0 & \frac{1-\rho_1}{|1-\rho_1|^3} & \frac{1-\rho_2}{|1-\rho_2|^3} \\ \frac{1-\rho_2}{|1-\rho_2|^3} & 0 & \frac{1-\rho_1}{|1-\rho_1|^3} \\ \frac{1-\rho_1}{|1-\rho_1|^3} & \frac{1-\rho_2}{|1-\rho_2|^3} & 0 \end{pmatrix},$$

$$\tilde{B}=\begin{pmatrix} \frac{1}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\ \frac{1}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\ \frac{1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \end{pmatrix},$$

$$\tilde{B}^*=\begin{pmatrix} \frac{1}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} \\ \frac{1}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} \\ \frac{1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{1}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} \end{pmatrix},$$

$$\tilde{D}=\begin{pmatrix} \frac{\rho_0}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_2}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\ \frac{\rho_2}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_0}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\ \frac{\rho_1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_2}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_0}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \end{pmatrix}$$

and

$$\tilde{D}^*=\begin{pmatrix} \frac{\rho_0}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_2}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} \\ \frac{\rho_2}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_0}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} \\ \frac{\rho_1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_2}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} & \frac{\rho_0}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} \end{pmatrix}.$$

Here, by the definition of ρ_l, it's not hard to see that $\rho_0 = e^{2\pi i*0/3} = 1, \rho_1 = e^{2\pi i*1/3} = e^{2\pi i/3}$ and $\rho_2 = e^{2\pi i*2/3} = e^{4\pi i/3}$. It is easy to verify that any one of A_1, B, B^*, D and D^* is a circulant matrix. Moreover, for the five matrices, we have the following lemmas.

Lemma 2.4. [16, Corollary] *For the eigenvalues of circulant matrix A_1, we have $\lambda_2(A_1) = 0$.*

Lemma 2.5. [24, Lemma 2.3] *Let E, F be any two circulant matrices, then matrices $E+F$ and $E-F$ are circulant. Moreover, the eigenvalues of $E+F$ and $E-F$ are $\lambda_j(E)+\lambda_j(F)$ and $\lambda_j(E)-\lambda_j(F)$, respectively.*

Lemma 2.6. *For the eigenvalues of circulant matrices $\tilde{B}$, $\tilde{B}^*$, $\tilde{D}$ and $\tilde{D}^*$, we have $\lambda_2(\tilde{B}^*) = \lambda_2(\tilde{D})$ and $\lambda_2(\tilde{D}^*) = \lambda_2(\tilde{B})$.*

Proof. Using the equalities that $\rho_4 = \rho_1^4 = e^{2\pi i*4/3} = e^{2\pi i/3} = \rho_1$, $\rho_1^2 = \rho_2$, $\cos(4\pi/3-\theta) = \cos(2\pi/3+\theta)$, $\cos(2\pi/3-\theta) = \cos(4\pi/3+\theta)$ where $\theta \in [0, 2\pi)$ and Lemma 2.1, we can calculate the second eigenvalues of circulant matrices $\tilde{B}$, $\tilde{B}^*$, $\tilde{D}$ and $\tilde{D}^*$ as follows.

$$\begin{aligned}\lambda_2(\tilde{B}) &= \sum_{1\le k\le 3}\frac{1}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}\rho_{2-1}^{k-1} \\ &= \frac{1}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_1}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_2}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\ &= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}}+\frac{\rho_1}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}} \\ &\quad +\frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}},\end{aligned}$$

$$\begin{aligned}\lambda_2(\tilde{B}^*) &= \sum_{1\le k\le 3}\frac{1}{(|\rho_{k-1}-re^{i\theta}|^2+h^2)^{\frac{3}{2}}}\rho_{2-1}^{k-1}\\ &= \frac{1}{(|\rho_0-re^{i\theta}|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_1}{(|\rho_1-re^{i\theta}|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_2}{(|\rho_2-re^{i\theta}|^2+h^2)^{\frac{3}{2}}}\\ &= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}}+\frac{\rho_1}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}-\theta)]^{\frac{3}{2}}}\\ &\quad+\frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}-\theta)]^{\frac{3}{2}}},\end{aligned}$$

$$\begin{aligned}\lambda_2(\tilde{D}) &= \sum_{1\le k\le 3}\frac{\rho_{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}\rho_{2-1}^{k-1}\\ &= \frac{1}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_2}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_4}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}\\ &= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}}+\frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}}\\ &\quad+\frac{\rho_4}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}}=\lambda_2(\tilde{B}^*)\end{aligned}$$

and

$$\begin{aligned}\lambda_2(\tilde{D}^*) &= \sum_{1\le k\le 3}\frac{\rho_{k-1}}{(|\rho_{k-1}-re^{i\theta}|^2+h^2)^{\frac{3}{2}}}\rho_{2-1}^{k-1}=\sum_{1\le k\le 3}\frac{\rho_2^{k-1}}{(|\rho_{k-1}-re^{i\theta}|^2+h^2)^{\frac{3}{2}}}\\ &= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}}+\frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}-\theta)]^{\frac{3}{2}}}\\ &\quad+\frac{\rho_4}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}-\theta)]^{\frac{3}{2}}}\\ &= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}}+\frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}}\\ &\quad+\frac{\rho_1}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}}=\lambda_2(\tilde{B}).\quad\square\end{aligned}$$

□

Furthermore, we require the following lemmas.

Lemma 2.7. *For the eigenvalues of circulant matrices $\tilde{B}$ and $\tilde{D}$, $\lambda_3(\tilde{B})=\lambda_2(\tilde{D})$ and $\bar{\lambda}_2(\tilde{B})=\lambda_3(\tilde{B})$ where $\bar{\lambda}_2(\tilde{B})$ is the complex conjugate of $\lambda_2(\tilde{B})$.*

Proof. Using Lemma 2.1 and the definitions of circulant matrices, we have

$$\begin{aligned}\lambda_3(\tilde{B}) &= \sum_{1\le k\le 3}\frac{\rho_2^{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}\\ &= \frac{1}{(|r\rho_0e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_2}{(|r\rho_1e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}+\frac{\rho_2^2}{(|r\rho_2e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}\\ &= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}}+\frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}}\\ &\quad+\frac{\rho_2^2}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}}.\end{aligned}$$

From the equalities $\rho_1\rho_1 = \rho_2$ and $\rho_2\rho_1^2 = e^{2\pi i*2/3} * e^{2\pi i*2/3} = e^{2\pi i*4/3} = \rho_2^2$, we have

$$\begin{aligned}
\lambda_2(\tilde{D}) &= \sum_{1\le k\le 3} \frac{\rho_{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}\rho_1^{k-1} \\
&= \frac{1}{(|r\rho_0 e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} + \frac{\rho_1\rho_1}{(|r\rho_1 e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} + \frac{\rho_2\rho_1^2}{(|r\rho_1 e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\
&= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}} + \frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}} \\
&\quad + \frac{\rho_2^2}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}} = \lambda_3(\tilde{B}).
\end{aligned}$$

Moreover, by performing a comparable calculation, we can derive that

$$\begin{aligned}
\bar{\lambda}_2(\tilde{B}) &= \overline{\sum_{1\le k\le 3} \frac{\rho_1^{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}}} = \sum_{1\le k\le 3} \frac{\rho_{-1}^{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\
&= \sum_{1\le k\le 3} \frac{\rho_2^{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} = \lambda_3(\tilde{B}). \ \Box
\end{aligned}$$

□

Lemma 2.8. *For any $\theta \in (0, 2\pi)$, we have $\lambda_2(\tilde{B}) \neq re^{i\theta}\bar{\lambda}_2(\tilde{B})$, where $r > 0$.*

Proof. From the proof of Lemma 2.7, we have

$$\begin{aligned}
\lambda_2(\tilde{B}) &= \sum_{1\le k\le 3} \frac{\rho_1^{k-1}}{(|r\rho_{k-1}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} = \frac{1}{(|r\rho_0 e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} + \frac{\rho_1}{(|r\rho_1 e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\
&\quad + \frac{\rho_2}{(|r\rho_2 e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} \\
&= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}} + \frac{\rho_1}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}} \\
&\quad + \frac{\rho_2}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}} \\
&= \frac{1}{[1+h^2+r^2-2r\cos\theta]^{\frac{3}{2}}} + \frac{-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{[1+h^2+r^2-2r\cos(\frac{2\pi}{3}+\theta)]^{\frac{3}{2}}} \\
&\quad + \frac{\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}}{[1+h^2+r^2-2r\cos(\frac{4\pi}{3}+\theta)]^{\frac{3}{2}}}.
\end{aligned} \tag{5}$$

Let $\lambda_2(\tilde{B}) = \mu_1 + i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$. In what follows, we prove $\lambda_2(\tilde{B}) \neq re^{i\theta}\bar{\lambda}_2(\tilde{B})$ by contradiction argument, and we assume that $\lambda_2(\tilde{B}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$ holds. Thus, if $\theta = \pi$, then $u_1 = -ru_1$ and

$$\cos(\frac{2\pi}{3}+\theta) - \cos(\frac{4\pi}{3}+\theta) = 0,$$

which implies that $u_1 \neq 0$. So, $r = -1$, which contradicts with $r > 0$. So $\theta \neq \pi$, and then $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Moreover, we have

$$\cos(\frac{2\pi}{3}+\theta) - \cos(\frac{4\pi}{3}+\theta) = -2\sin\frac{\pi}{3}\sin\theta = -\sqrt{3}\sin\theta \neq 0. \tag{6}$$

□

Combining (5)–(6) and $\lambda_2(\tilde{B}) = \mu_1 + i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$, we have $\mu_2 \neq 0$. Thus, $\lambda_2(\tilde{B}) \neq 0$. On the one hand, by $\lambda_2(\tilde{B}) = \mu_1 + i\mu_2 \neq 0$ and $\lambda_2(\tilde{B}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$, where $r > 0$, we have

$$\begin{cases} |\frac{\lambda_2(\tilde{B})}{\bar{\lambda}_2(\tilde{B})}| = |\frac{\mu_1 + i\mu_2}{\mu_1 - i\mu_2}| = r, \\ u_1^2 + u_2^2 \neq 0. \end{cases}$$

Hence,

$$r = |\frac{(\mu_1 + i\mu_2)^2}{u_1^2 + u_2^2}| = \frac{|(\mu_1^2 - \mu_2^2) + 2iu_1u_2|}{u_1^2 + u_2^2} = \frac{\sqrt{(\mu_1^2 - \mu_2^2)^2 + 4\mu_1^2\mu_2^2}}{u_1^2 + u_2^2} = \frac{u_1^2 + u_2^2}{u_1^2 + u_2^2} = 1.$$

On the other hand, with the aid of $\theta \in (0, \pi) \cup (\pi, 2\pi)$, we have $\sin\theta \neq 0$. Then, employing (5)–(6) and $\lambda_2(\tilde{B}) = \mu_1 + i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$, we conclude that

$$\begin{cases} \text{if} \quad \sin\theta > 0, \quad \text{then} \quad \cos(\frac{2\pi}{3} + \theta) < \cos(\frac{4\pi}{3} + \theta), \quad \text{which implies that} \quad \mu_2 < 0; \\ \text{if} \quad \sin\theta < 0, \quad \text{then} \quad \cos(\frac{2\pi}{3} + \theta) > \cos(\frac{4\pi}{3} + \theta), \quad \text{which implies that} \quad \mu_2 > 0. \end{cases} \tag{7}$$

Note that $\lambda_2(\tilde{B}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$, then thanks to direct computation, there is

$$\mu_1 + \mu_2 i = r(\cos\theta + i\sin\theta)(\mu_1 - \mu_2 i) = r(\mu_1\cos\theta + \mu_2\sin\theta) + r(\mu_1\sin\theta - \mu_2\cos\theta)i.$$

Combining $r = 1$, one computes that

$$\begin{cases} \mu_1\cos\theta + \mu_2\sin\theta = \mu_1, \\ \mu_1\sin\theta - \mu_2\cos\theta = \mu_2, \end{cases} \tag{8}$$

which implies that

$$\begin{cases} \mu_1(1 - \cos\theta) = \mu_2\sin\theta, \\ \mu_2(1 + \cos\theta) = \mu_1\sin\theta. \end{cases} \tag{9}$$

In the following, we will demonstrate that neither of the two cases in (7) can occur.

Case 1. We assume that $\sin\theta > 0$ holds. In this case, by the first part of (7), there is $\mu_2 < 0$. Then, inserting $\sin\theta > 0$ and $\mu_2 < 0$ into the first part of (9), we have $\mu_1(1 - \cos\theta) < 0$. Thus, it follows from $\theta \in (0, \pi) \cup (\pi, 2\pi)$ that $\mu_1 < 0$. In fact, inserting $\sin\theta > 0$ into (6), we have $\cos(2\pi/3 + \theta) < \cos(4\pi/3 + \theta)$. Combining $\lambda_2(\tilde{B}) = \mu_1 + i\mu_2$ and (5), we have $\mu_1 > 0$, which contradicts with $\mu_1 < 0$. Hence, we deduce a contradiction.

Case 2. We assume that $\sin\theta < 0$ holds. In this case, combining $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and the second part of (7), we have $\pi < \theta < 2\pi$, $\cos(2\pi/3 + \theta) > \cos(4\pi/3 + \theta)$ and $u_2 > 0$. Combining $\sin\theta < 0$, $\pi < \theta < 2\pi$ and the first part of (9), one computes that $u_1 < 0$. Next, we divide the procedure of Case 2 into the following two parts.

Firstly, when $\cos(2\pi/3 + \theta) \leq 0$, note that

$$\cos(\frac{2\pi}{3} + \theta) + \cos(\frac{4\pi}{3} + \theta) = 2\cos\frac{\pi}{3}\cos(\pi + \theta) = -\cos\theta,$$

and $\cos(4\pi/3 + \theta) < \cos(2\pi/3 + \theta) \leq 0$, one computes that $\cos\theta > 0$. Inserting $\cos(4\pi/3 + \theta) < \cos(2\pi/3 + \theta) \leq 0$ and $\cos\theta > 0$ into (5), we have $u_1 > 0$, which contradicts with $u_1 < 0$.

Secondly, when $\cos(2\pi/3 + \theta) > 0$, using $\pi < \theta < 2\pi$, we have $5\pi/3 < 2\pi/3 + \theta < 5\pi/2$, which implies that $\pi < \theta < 11\pi/6$. Then, employing (9) along with $\mu_1 < 0$, $\mu_2 > 0$ and $\pi < \theta < 11\pi/6$, it is not difficult to verify that

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{u_2^2}{u_1^2}. \tag{10}$$

On the one hand, employing (8), direct computation shows us that

$$u_1u_2\cos\theta + u_2^2\sin\theta = u_1^2\sin\theta - u_1u_2\cos\theta,$$

which gives us information that

$$2u_1u_2\cos\theta = (u_1^2 - u_2^2)\sin\theta. \tag{11}$$

Then, inserting $\mu_1 < 0$, $\mu_2 > 0$, $\sin\theta < 0$ and $\cos\theta < 0$ into (11), one computes that $u_2^2 > u_1^2$, and then $\mu_2 > -\mu_1 > 0$. Combining the second part of (9), there is

$$0 = \mu_2(1+\cos\theta) - \mu_1\sin\theta > -u_1(1+\cos\theta+\sin\theta) = -u_1[1+\sqrt{2}\cos(\theta - \frac{\pi}{4})]. \tag{12}$$

On the other hand, inserting $u_2^2 > u_1^2$ and $\pi < \theta < 11\pi/6$ into (10), we have $\cos\theta < 0$, which implies that $\pi < \theta < 3\pi/2$, and then $3\pi/4 < \theta - \pi/4 < 5\pi/4$. Hence, $\cos(\theta - \pi/4)) > -\sqrt{2}/2$. Then, with $-u_1 > 0$ and the aid of (12), there is

$$0 = \mu_2(1+\cos\theta) - \mu_1\sin\theta > -u_1(1+\cos\theta+\sin\theta) > -u_1[1+\sqrt{2}\cos(\theta - \frac{\pi}{4})] > 0,$$

and it is impossible. Therefore, Cases 1-2 cannot happen. Thus, for any $\theta \in (0, 2\pi)$ and any $r > 0$, $\lambda_2(\tilde{B}) \neq re^{i\theta}\bar{\lambda}_2(\tilde{B})$ is impossible.

By now, we have completed the proof of Lemma 2.8. □

3. Proof of Theorem 1.1

From (1) and (3), it follows that the spatial central configurations satisfy the following equations for any $k \in \{1,2,3\}$.

$$\begin{cases} \sum\limits_{\substack{j\neq k\\ 1\le j\le 3}} \frac{(\rho_j-\rho_k,0)m_jm_k}{|\rho_j-\rho_k|^3} + \sum\limits_{1\le j\le 3} \frac{(r\rho_je^{i\theta}-\rho_k,h)m_{3+j}m_k}{(|r\rho_je^{i\theta}-\rho_k|^2+h^2)^{\frac{3}{2}}} = -\lambda m_k(\rho_k - c_0,\, -h_0), \\ \sum\limits_{1\le j\le 3} \frac{(\rho_j-r\rho_{3+k}e^{i\theta},-h)}{(|\rho_j-r\rho_{3+k}e^{i\theta}|^2+h^2)^{\frac{3}{2}}} m_jm_{3+k} + \sum\limits_{\substack{j\neq k\\ 1\le j\le 3}} \frac{(r\rho_{3+j}e^{i\theta}-r\rho_{3+k}e^{i\theta},0)}{(|r\rho_{3+j}e^{i\theta}-r\rho_{3+k}e^{i\theta}|^2+0)^{\frac{3}{2}}} m_{3+j}m_{3+k} \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\lambda m_{3+k}(r\rho_{3+k}e^{i\theta} - c_0,\, h - h_0), \end{cases}$$

where the center of masses $x_0 := (c_0, h_0)$. Then, employing $\rho_{3\pm j} = \rho_{\pm j}$ with $j \in \{1,2,3\}$, we have

$$\begin{cases} \sum\limits_{\substack{j\neq k\\ 1\le j\le 3}} \frac{(\rho_{j-k}-1,0)m_j}{|\rho_{j-k}-1|^3} + \sum\limits_{1\le j\le 3} \frac{(r\rho_{j-k}e^{i\theta}-1,h)m_{3+j}}{(|r\rho_{j-k}e^{i\theta}-1|^2+h^2)^{\frac{3}{2}}} = -\lambda(1 - c_0\rho_{-k},\, -h_0), \\ \sum\limits_{1\le j\le 3} \frac{(\rho_{j-k}-re^{i\theta},-h)}{(|\rho_{j-k}-re^{i\theta}|^2+h^2)^{\frac{3}{2}}} m_j + \frac{e^{i\theta}}{r^2} \sum\limits_{\substack{j\neq k\\ 1\le j\le 3}} \frac{(\rho_{j-k}-1,0)}{|\rho_{j-k}-1|^3} m_{3+j} = -\lambda(re^{i\theta} - c_0\rho_{-k},\, h - h_0). \end{cases} \tag{13}$$

After combining (4) and (13), using the definitions of matrices A_1, $\tilde{B}$, $\tilde{B}^*$, $\tilde{D}$, and $\tilde{D}^*$, we have

$$\begin{cases} -A_1m + (re^{i\theta}\tilde{D} - \tilde{B})M = -\lambda\nu_1 + c_0\lambda\nu_3, \\ (\tilde{D}^* - re^{i\theta}\tilde{B}^*)m - \frac{e^{i\theta}}{r^2}A_1M = -\lambda re^{i\theta}\nu_1 + c_0\lambda\nu_3, \end{cases} \tag{14}$$

where $\widetilde{m} = (m_1, m_2, m_3)^T$ and $M = (m_4, m_5, m_6)^T$.

According to Lemma 2.2, there exist certain constants a_1, a_2, a_3, b_1, b_2 and b_3 such that

$$\begin{cases} \widetilde{m} = a_1\nu_1 + a_2\nu_2 + a_3\nu_3, \\ M = b_1\nu_1 + b_2\nu_2 + b_3\nu_3. \end{cases} \tag{15}$$

Using (14) and (15), we can conclude that

$$\begin{cases} -A_1(\sum\limits_{1\le j\le 3} a_j\nu_j) + (re^{i\theta}\tilde{D} - \tilde{B})(\sum\limits_{1\le j\le 3} b_j\nu_j) = -\lambda\nu_1 - c_0\lambda\nu_3, \\ (\tilde{D}^* - re^{i\theta}\tilde{B}^*)(\sum\limits_{1\le j\le 3} a_j\nu_j) - \frac{e^{i\theta}}{r^2}A_1(\sum\limits_{1\le j\le 3} b_j\nu_j) = -\lambda re^{i\theta}\nu_1 + c_0\lambda\nu_3. \end{cases} \tag{16}$$

Employing (16) and Lemmas 2.1–2.3, it becomes evident that

$$\begin{cases} -3\lambda_2(A_1)a_2 + 3\lambda_2(re^{i\theta}\tilde{D} - \tilde{B})b_2 = 0, \\ 3\lambda_2(\tilde{D}^* - re^{i\theta}\tilde{B}^*)a_2 - 3\frac{e^{i\theta}}{r^2}\lambda_2(A_1)b_2 = 0. \end{cases}$$

In addition, using Lemmas 2.4–2.5, we can show that

$$\begin{cases} [\lambda_2(re^{i\theta}\tilde{D}) - \lambda_2(\tilde{B})]b_2 = 0, \\ [\lambda_2(\tilde{D}^*) - \lambda_2(re^{i\theta}\tilde{B}^*)]a_2 = 0. \end{cases} \tag{17}$$

By the form of eigenvalues λ_j in Lemma 2.1, we have $\lambda_2(re^{i\theta}\tilde{D}) = re^{i\theta}\lambda_2(\tilde{D})$ and $\lambda(re^{i\theta}\tilde{B}) = re^{i\theta}\lambda_2(\tilde{B})$. Then, thanks to Lemma 2.6 and (17), we have

$$\begin{cases} [re^{i\theta}\lambda_2(\tilde{D}) - \lambda_2(\tilde{B})]b_2 = 0, \\ [\lambda_2(\tilde{B}) - re^{i\theta}\lambda_2(\tilde{D})]a_2 = 0. \end{cases} \tag{18}$$

Next, we will now break down the remaining proof of Theorem 1.1 into three separate cases.

Case 3. $a_2 \neq 0$ and $b_2 \neq 0$.

Due to Lemma 2.7, we have $\lambda_2(\tilde{D}) = \lambda_3(\tilde{B}) = \bar{\lambda}_2(\tilde{B})$. Then, it follows from the second equation of (18) and $a_2 \neq 0$ that $\lambda_2(\tilde{B}) = re^{i\theta}\lambda_2(\tilde{D}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$. On the other hand, if $\theta \in (0, 2\pi)$, then Lemma 2.8 shows us that $\lambda_2(\tilde{B}) \neq re^{i\theta}\bar{\lambda}_2(\tilde{B})$, which contradicts with $\lambda_2(\tilde{B}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$, So $\theta = 0$. Then, it follows from $h > 0$ and the Theorem 1.2 of page 4822 in [18] that $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$.

Case 4. $b_2 = 0$. In this case, from the second equation of (15), we have $M = b_1\nu_1 + b_3\nu_3$.

Assume that

$$\begin{cases} b_1 = b_1^{(1)} + ib_1^{(2)}, \text{ where } b_1^{(1)} \in \mathbb{R} \text{ and } b_1^{(2)} \in \mathbb{R}, \\ b_3 = b_3^{(1)} + ib_3^{(2)}, \text{ where } b_3^{(1)} \in \mathbb{R} \text{ and } b_3^{(2)} \in \mathbb{R}. \end{cases} \tag{19}$$

Then, combining (19), $\widetilde{m} = (m_1, m_2, m_3)^T \in (\mathbb{R}, \mathbb{R}, \mathbb{R})^T$, $M = (m_4, m_5, m_6)^T \in (\mathbb{R}, \mathbb{R}, \mathbb{R})^T$, $\nu_1 = (1, 1, 1)^T$ and $\nu_3 = (\rho_2, \rho_4, \rho_6)^T = (\rho_2, \rho_1, \rho_3)^T$, we have

$$\begin{cases} (b_1^{(1)} + ib_1^{(2)}) + (b_3^{(1)} + ib_3^{(2)})\rho_2 \in \mathbb{R}, \\ (b_1^{(1)} + ib_1^{(2)}) + (b_3^{(1)} + ib_3^{(2)})\rho_1 \in \mathbb{R}, \\ (b_1^{(1)} + ib_1^{(2)}) + (b_3^{(1)} + ib_3^{(2)})\rho_3 \in \mathbb{R}. \end{cases}$$

Therefore,

$$\begin{cases} b_1^{(2)} - \frac{\sqrt{3}}{2}b_3^{(1)} - \frac{1}{2}b_3^{(2)} = 0, \\ b_1^{(2)} + \frac{\sqrt{3}}{2}b_3^{(1)} - \frac{1}{2}b_3^{(2)} = 0, \\ b_1^{(2)} + b_3^{(2)} = 0, \end{cases} \tag{20}$$

and we can deduce from (20) that $b_1^{(2)} = b_3^{(1)} = b_3^{(2)} = 0$. Therefore, $b_1 = b_1^{(1)} \in \mathbb{R}$ and $b_3 = 0$. This implies that when $b_2 = 0$, $m_4 = m_5 = m_6$.

Next, for the case where $b_2 = 0$, we will prove that $m_1 = m_2 = m_3$.

On one hand, if $a_2 = 0$, then according to (15), $\widetilde{m} = a_1\nu_1 + a_3\nu_3$. Similar to the above procedure of $M = b_1\nu_1 + b_3\nu_3$, we obtain $m_1 = m_2 = m_3$.

On the other hand, if $a_2 \neq 0$, then by the second equation of (18), we can conclude that $\lambda_2(\tilde{B}) = re^{i\theta}\lambda_2(\tilde{D})$, then using Lemma 2.7, we obtain $\lambda_2(\tilde{B}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$. Thus, by the same procedure used in Case 3, we deduce that $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$. Therefore, we complete the proof of Case 4.

Case 5. $a_2 = 0$. In this case, we can combine the first part of (15) to obtain $\widetilde{m} = a_1\nu_1 + a_3\nu_3$, and following the proof of $m_4 = m_5 = m_6$ in Case 4, we can conclude that $m_1 = m_2 = m_3$.

In the case where $a_2 = 0$, we will prove that $m_4 = m_5 = m_6$ as the following.

On one hand, if $b_2 = 0$, then from the first part of (15), we have $M = b_1\nu_1 + b_3\nu_3$. Therefore, $m_4 = m_5 = m_6$.

On the other hand, if $b_2 \neq 0$, then by Lemma 2.7 and the first equation of (18), we have $\lambda_2(\tilde{B}) = re^{i\theta}\lambda_2(\tilde{D}) = re^{i\theta}\lambda_3(\tilde{B}) = re^{i\theta}\bar{\lambda}_2(\tilde{B})$. Then, similar to the procedure of Case 3, we have $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$, which implies that the proof of Case 5 is completed. □

4. Conclusions

It is well-known that for the Lagrange equilateral-triangle central configuration of Newtonian 3-body problem formed by an equilateral triangle, the masses located at the vertices of the equilateral triangle may not be equal. But for planar twisted configurations formed by two concentric equilateral triangles with any twist angle θ, the authors of the paper [24] found that if the masses in each separate equilateral triangle are not equal, then the six masses cannot form a central configuration. Subsequently, for the spatial twisted configurations formed by two parallel equilateral triangles, when the twist angle is $\theta = 0$, Theorem 1.2 of [18] implies that there is no central configuration with unequal masses located at the vertices of each separate equilateral triangle. Now, we prove that if $\theta \in [0, 2\pi]$, then there is no spatial twisted central configuration with unequal masses located at the vertices of each separate equilateral triangle.

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